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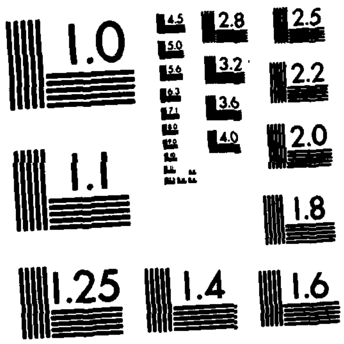

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PERIODIC SOLUTIONS OF NON-DISSIPATIVELY PERTURBED WAVE EQUATIONS IN SEVERAL SPACE VARIABLES

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PERIODIC SOLUTIONS OF NON-DISSIPATIVELY PERTURBED WAVE EQUATIONS IN SEVERAL SPACE VARIABLES

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ABSTRACT

We consider the perturbed wave equation:

$$(*)\qquad \begin{cases} u_{tt} - \Delta u + \varepsilon f(x,t,u) = 0 & \text{for } x \in \Omega \subset \mathbb{R}^n \\ & t \in \mathbb{R} \\ u(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R} & \end{cases}$$

where f depends monotonically on u and is periodic in t. Periodic solutions are constructed for ε sufficiently small under the following hypotheses:

(i) Ω is a domain such that the eigenvalues $\{\lambda_n\}$ of $-\Delta$ form a rational sequence - i.e.

$$\lambda_k/\lambda_0 \in \mathbb{Q} \quad .$$

(ii) The period of f is rationally related to the periods of the free vibrations for the wave equation, namely $\frac{2\pi}{\sqrt{\lambda_n}}$

(iii) f depends monotonically on u and is sufficiently smooth - $f \in C^r$ leads to a solution in H^ρ for $r > \frac{n+1}{2}$ and all $\rho < r$.

This generalizes a result of Rabinowitz [6] to more than one space variable.

AMS (MOS) Subject Classifications: 35B10, 35L05

Key Words: Multi-dimensional non-linear wave equation; monotone perturbation

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

pi-

In [6], Rabinowitz proved the existence of 2π-periodic solutions of ~~the~~ a one-dimensional non-linear wave equation

$$(**)\qquad \begin{cases} u_{tt} - u_{xx} + \epsilon f(x,t,u) = 0 \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

pi-

for ϵ sufficiently small and f 2π-periodic in t, monotone in u, and sufficiently smooth. This answered a long-standing open question and suggested that monotone methods could be used to overcome solvability problems in bifurcation situations with infinite dimensional kernels. In this paper the methods of Rabinowitz [6] are extended to higher space dimensions, indicating that the special properties of the one-dimensional wave equation are not essential for that result. What remains crucial are hypotheses of rationality in the relations between the time period and the periods of the free vibrations for the wave equation, so that the inverse of the wave operator remains bounded on the complement of the null space. The other crucial factor is the assumption that the non-linearity depends monotonically on u, which enables us to solve for the piece of the solution lying in the (possibly infinite dimensional) null space of the wave operator.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

PERIODIC SOLUTIONS OF NON-DISSIPATIVELY PERTURBED WAVE EQUATIONS IN SEVERAL SPACE VARIABLES

Robert L. Sachs

1. Introduction

We consider the partial differential equation

$$(*) \qquad u_{tt}(x,t) - \Delta u(x,t) + \varepsilon f(x,t,u(x,t)) = 0$$

for a function $u(x,t)$ where x is an n-vector, t is a scalar, and f has period $2\pi/\omega$ in t, and seek solutions satisfying the periodicity and boundary conditions:

$$(1) \qquad \begin{cases} u(x,t+2\pi/\omega) = u(x,t) \\ u(x,t) = 0 \quad \text{for all } t \text{ whenever } x_j = 0 \text{ or } x_j = L_j \\ \qquad\qquad\qquad \text{for any integer } 1 \leq j \leq n \ . \end{cases}$$

In other words, the x-variables range over the interior of a rectangular parallelopiped $\Omega \subset \mathbb{R}^n$ and we impose the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. For reasons discussed below, we restrict our attention further to those Ω for which the side lengths L_j are all rational multiples of one another and for which the eigenvalues of $-\Delta$ on Ω (which form a rationally related sequence) are rational multiples of ω^2. Any L^2-function $v(x,t)$ satisfying the boundary and periodicity conditions (1) above may be represented by a Fourier series of the form:

$$(2) \qquad v(x,t) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}_+^n} a_{jk} e^{ik\omega t} \sin\frac{\pi j_1 x_1}{L_1} \cdots \sin\frac{\pi j_n x_n}{L_n}$$

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where the multi-index $j \in Z_+^n$ denotes the n-tuple of positive integers $(j_1, j_2, \ldots, j_n)$. If $\Box v = v_{tt} - \Delta v$ is in L^2, we have:

$$(3) \qquad \Box v = \sum_{k \in Z} \sum_{j \in Z_+^n} \left[-k^2\omega^2 + \pi^2\left(\frac{j_1^2}{L_1^2} + \ldots + \frac{j_n^2}{L_n^2}\right) \right] a_{jk} e^{ik\omega t} \sin \frac{\pi j_1 x_1}{L_1} \ldots \sin \frac{\pi j_n x_n}{L_n} .$$

Our rationality restrictions ensure that the 'Fourier multiplier' in brackets above, $-k^2\omega^2 + \pi^2\left(\frac{j_1^2}{L_1^2} + \ldots + \frac{j_n^2}{L_n^2}\right)$, although it may vanish for infinitely many choices of j and k, is bounded away from 0 for all other choices of j and k. For more general domains in several space variables, $-\Delta$ typically has a less well-behaved (in the sense of rational multiples) sequence of eigenvalues and this leads to number - theoretic difficulties of 'small divisors' when inverting the linear wave operator $\Box$. Even in one space variable, these difficulties have not, to the best of our knowledge, been overcome to date on problems of this type. In the case where $-\Delta$ with any given boundary conditions has such a spectrum, our results hold.

For most of the paper, we will consider $n = 2$ and Ω a square with side length π. This is for notational and expository ease and the modifications required for the general case will be indicated in Section 5 below. If x, y denote the variables in the square, then $-\Delta = -\partial_x^2 - \partial_y^2$ with Dirichlet boundary conditions has eigenvalues $\lambda = j^2 + \ell^2$ where j, ℓ are positive integers. Thus 2 is the lowest eigenvalue and all the eigenvalues are integers. A natural choice of ω is then $\sqrt{2}$. We discuss this case and again, indicate the necessary modifications for the general case of ω^2 rational at the end of the paper.

Our results and methods are the direct generalizations of Rabinowitz [6, Part I]. We give proofs for the sake of completeness but acknowledge our extreme reliance on the ideas and method of [6].

Before stating our results, we introduce some notation. $C^\infty(C_0^\infty)$ will denote the infinitely differentiable functions in (x,y,t), $\sqrt{2}\pi$ periodic in t (with support in the set $0 < x,y < \pi$). Let $H^0 = L^2$ be the completion of C^∞ with respect to

$$|\phi|_0^2 = |\phi|^2 = \int_0^{\sqrt{2}\pi}\int_0^\pi\int_0^\pi |\phi(x,y,t)|^2 dxdydt$$

and H^s be the completion of C^∞ with respect to

$$|\phi|_s^2 = \sum_{|\sigma|=0}^{s} |D^\sigma\phi|^2 \quad \text{where} \quad \sigma = (\sigma_1,\sigma_2,\sigma_3)$$

$|\sigma| = \sigma_1 + \sigma_2 + \sigma_3$ with $D^\sigma = \dfrac{\partial^{|\sigma|}}{\partial x^{\sigma_1}\partial y^{\sigma_2}\partial t^{\sigma_3}}$ and let R denote the region $0 < x < \pi$, $0 < y < \pi$, $0 < t < \sqrt{2}\pi$. We denote by H_0^s the completion of C_0^∞ with respect to $|\ |_s$. These are all Hilbert spaces with inner products $(\ ,\ )_s$. Let C^r be the space of r times continuously differentiable functions of x,y,t, $\sqrt{2}$ periodic in t for $0 < x,y < \pi$. We denote the usual max norm in C^r by $\|\ \|_r$ - thus

$$\|\phi\|_r = \sum_{|\sigma|=0}^{r} \|D^\sigma\phi\| \quad \text{where} \quad \|\phi\| = \sup_R |\phi(x,y,t)| \ .$$

Our main result is the following:

<u>Theorem 1</u>. If $f \in C^k$ and $\sqrt{2}\pi$ periodic in t, with $f_u > \beta > 0$, then, for $|\varepsilon|$ sufficiently small, the partial differential equation

$$(*) \qquad u_{tt} - u_{xx} - u_{yy} + \varepsilon f(x,y,t,u) = 0$$

has a solution $u \in H_0^\rho$ such that $u(x,y,t + \sqrt{2}\pi) = u(x,y,t)$, where $\rho < k$. In particular, if $k > 7/2$, u is C^2 and hence a classical solution of (*).

The basic idea is to utilize the splitting of the Hilbert spaces H^s into two pieces - the part intersecting the null space of the linear wave operator and its orthogonal complement. Here the null space is infinite-dimensional, so this may be viewed as a Liapunov-Schmidt procedure in a degenerate case. Thus we rewrite (*) as the pair of relations with $u = v + \varepsilon w$ where $v \in N$, $w \in N^\perp$

$$(4) \qquad \begin{cases} \Box w + f(x,y,t,v+\varepsilon w) = 0 , \\ f(x,y,t,v+\varepsilon w) \in R(\Box) = N(\Box)^\perp . \end{cases}$$

The second relation is the 'bifurcation equation' for this problem, since at $\varepsilon = 0$, (*) becomes

$$(5) \qquad \Box u = 0 ,$$

which has infinitely many periodic solutions. As in [6], the bifurcation point may be found by solving a convex variational problem, which we do in Section 2 below. In Section 3, solutions of the linear wave equation are discussed. We note that, unlike the case of one space variable, no derivatives are gained. For our problem, this is not a serious difficulty. The full non-linear problem is treated in Section 4, while in Section 5 we discuss the modifications needed to handle more general periods, unequal side lengths, and higher space dimensions. One technical argument needed in the proof of Theorem 4 is relegated to the appendix.

2. The Bifurcation Equation

We begin by considering the null space of the wave operator, $\square$, with the boundary and periodicity conditions (1). It is easy to see that if N is the L^2-closure of the smooth, real solutions of $\square u = 0$ with the boundary and periodicity conditions (1) -

$$\begin{cases} u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 \\ u(x,y,t+\sqrt{2}\pi) = u(x,y,t) \end{cases}$$

then $N \equiv \{\phi(x,y,t) \in L^2:$

$$\phi(x,y,t) = \sum_{2k^2=j^2+\ell^2} \sum_{j>0} \sum_{\ell>0} a_{jk\ell} e^{\sqrt{2}ikt} \sin jx \sin \ell y$$

$$\text{with } \sum\sum\sum |a_{jk\ell}|^2 < \infty \text{ and } a_{j,-k,\ell} = \overline{a_{j,k,\ell}}\} .$$

To solve (*), split u into pieces $v \in N$ and $\varepsilon w \in N^{\perp}$. Then $u = v(x,y,t,\varepsilon) + \varepsilon w(x,y,t,\varepsilon)$ with $\square u + \varepsilon f = 0$ implies that we must have $f(x,y,t,v+\varepsilon w) \in N^{\perp}$. Suppose $w \in H^r$ is given. We shall solve:

(6) $$f(x,y,t,v+\varepsilon w) \in N^{\perp}$$

by variational methods as in [6].

Remark. In one space variable, $\phi \in N \iff \phi = p(x+t) - p(-x+t)$ $p \in L^2(S^1)$. For several space variables, ϕ is not so simply represented. In particular, the Diophantine equation $2k^2 = j^2 + \ell^2$ has many solutions besides the obvious sequence $j = \ell = \pm k = s$, $s = 1,2,3,\ldots$ such as $j = s$, $\ell = 7s$, $k = \pm 5s$ and $j = 17s$, $\ell = 31s$, $k = \pm 25s$. Nevertheless, for our use below, we represent $\phi \in N$ as follows:

$$\phi(x,y,t) = \sum_{j>0} \sum_{\ell>0} \left(b_{j\ell} e^{i\sqrt{j^2+\ell^2}\, t} + \overline{b_{j\ell}}\, e^{-i\sqrt{j^2+\ell^2}\, t}\right) \sin jx \sin \ell y$$

$$= -\frac{1}{4} \sum_{j>0} \sum_{\ell>0} \left(b_{j\ell} e^{i\sqrt{j^2+k^2}\, t} + \overline{b_{j\ell}}\, e^{-i\sqrt{j^2+\ell^2}\, t}\right) [e^{i(jx+\ell y)} - e^{i(jx-\ell y)}$$

$$- e^{i(-jx+\ell y)} + e^{i(-jx-\ell y)}]$$

$$= \frac{1}{2} \sum_{j>0} \sum_{\ell>0} \mathrm{Re}\{b_{j\ell}[e^{i(-jx+\ell y+\sqrt{j^2+\ell^2}\, t)} + e^{i(jx-\ell y+\sqrt{j^2+\ell^2}\, t}]\}$$

$$-\frac{1}{2} \sum_{j>0} \sum_{\ell>0} \mathrm{Re}\{b_{j\ell}[e^{i(jx+\ell y+\sqrt{j^2+\ell^2}\, t} + e^{i(-jx-\ell y+\sqrt{j^2+\ell^2}\, t}]\} \equiv \phi^{(1)} - \phi^{(2)} .$$

Note that each term extends for x, y beyond the square and in this extended sense, each $\phi^{(j)}$ is symmetric about the origin in the x, y plane; also $\phi^{(1)}(-x,y,t) = \phi^{(2)}(x,y,t)$. Thus we have

$$M = \max_R (|\phi^{(1)}|, |\phi^{(2)}|) = \max_{\substack{0\le x\le\pi \\ -\pi\le y\le\pi \\ 0\le t\le\sqrt{2}\pi}} |\phi^{(1)}| = \max_{\substack{0\le x\le\pi \\ -\pi\le y\le\pi \\ 0\le t\le\sqrt{2}\pi}} |\phi^{(2)}| .$$

Also observe that if $\phi^{(1)}, \phi^{(2)} \in H^s$, they are orthogonal with respect to the H^s inner product for every s.

We shall prove the following:

<u>Theorem 2</u>. If $f(x,y,t,u) \in C^2$, $f_u \ge \beta > 0$ and w is given, $w \in H^2$, then there exists a unique $v \in N \cap H^2$ such that

$$\iiint_R f(x,y,t,v+w)\ \phi(x,y,t)\,dxdydt = 0$$

$$\text{for all } \phi \in N .$$

Moreover,

$$\beta/4 \sup_R |v(x,y,t)| \le \sup_R |f(x,y,t,w)|$$

$$\beta|D^1 v|_0 \le C_1[|f_t(x,y,t,v+w)|_0 + |f_u(x,y,t,v+w)w_t|_0]$$

$$\beta|D^2v|_0 < C_2(|f_{tt}(x,y,t,v+w)|_0 + 2|f_{ut}(x,y,t,v+w)(v_t+w_t)|_0$$

$$+ |f_{uu}(x,y,t,v+w)(v_t+w_t)^2|_0 + |f_u(x,y,t,v+w)w_{tt}|_0)$$

where C_1, C_2 are constants.

The proof of Theorem 2 is based on solving an approximate problem which is better behaved. Namely, consider $F(x,y,t,u)$, where $F_u = f$. Let N_K denote the subset of N defined by: $N_K = \{\phi \in N \mid |\phi|_2 < K\}$. N_K is compactly imbedded in N; in fact, the imbedding is a Hilbert-Schmidt operator (see Adams [1], p. 174 or Maurin [3]). We shall prove:

<u>Theorem 3</u>. There exists $v_K \in N_K$ minimizing the functional

$$\iiint_R F(x,y,t,\phi+w)\,dxdydt \quad \text{over all} \quad \phi \in N_K \; .$$

Moreover,

$$\begin{cases} \beta/4 \sup\limits_R |v_K(x,y,t)| < \sup\limits_R |f(x,y,t,w(x,y,t))| \\[2ex] \beta\Big(\sum\limits_{|\sigma|=1} |D^\sigma v_K|_0^2\Big)^{1/2} < C_1[|f_t(x,y,t,v_K+w)|_0 + |f_u(x,y,t,v_K+w)w_t|_0] \\[2ex] \beta\Big(\sum\limits_{|\sigma|=2} |D^\sigma v_K|_0^2\Big)^{1/2} < C_2[|f_{tt}(x,y,t,v_K+w)|_0 + 2|f_{ut}(x,y,t,v_K+w)(v_t+w_t)|_0 \\[2ex] \qquad\qquad + |f_{uu}(\cdot,\cdot,\cdot,v_K+w)(v_{K_t}+w_t)^2|_0 + |f_u(\cdot,\cdot,\cdot,v_K+w)w_{tt}|_0] \; . \end{cases}$$

<u>Proof of Theorem 3</u>: We show that a minimizer v_K exists, then use some particular admissible variations about v_K to derive the estimates above.

The functional $\int_R F(x,y,t,\phi+w)\,dxdydt$ is continuous on N_K in the H^0 topology (w is fixed and bounded and $\phi \in N_K$ implies ϕ is bounded). Hence by the compactness of N_K, there is a minimum which is taken on by some $v_K \in N_K$. We will show that , when K is sufficiently large, v_K is an interior minimum.

To this end, we consider variations $v_K + \eta\phi$ about v_K, where $v_K + \eta\phi \in N_K$ for $\eta < 0$ sufficiently small. Then we have:

$$0 \geq \iiint_R [F(x,y,t,v_K+w) - F(x,y,t,v_K+\eta\phi+w)]\,dxdydt$$

$$\equiv \iiint_R f(x,y,t,v_K+w+\eta\rho(x,y,t)\,\phi)(-\eta\phi)\,dxdydt$$

for some $|\rho| \leq 1$ by the mean value theorem. Dividing by $-\eta$ and letting $\eta \to 0$, $\eta\rho\phi \to 0$ uniformly the 'Euler equation' for our minimization problem becomes:

(7) $$\iiint_R f(x,y,t,v_K+w)\,\phi\,dxdydt \leq 0 \quad ,$$

for all admissible variations ϕ. If $|v_K+\eta\phi|_2^2 \leq K^2$, ϕ is an admissible variation $(v_K+\eta\phi \in N_K)$ so for $\eta < 0$ sufficiently small, it suffices to require that $(v_K,\phi)_2 > 0$.

One admissible variation is the following:

Let $$q(\lambda) \equiv \begin{cases} 0 & \text{if } |\lambda| \leq m \\ \lambda-m & \text{if } \lambda \geq m \\ \lambda+m & \text{if } \lambda \leq -m \end{cases} .$$

Writing $v_K \in N_K$ as $v_K^{(1)} - v_K^{(2)}$ by our previous remark, choosing $m = \frac{1}{2}M$, then

$$(v_K,q(v_K^{(1)}) - q(v_K^{(2)}))_2 = (v_K^{(1)},q(v_K^{(1)}))_2 + (v_K^{(2)},(v_K^{(2)}))_2$$

by the orthogonality of $v_K^{(1)}$, $v_K^{(2)}$ in H^2. Moreover,

$$(v_K^{(i)},q(v_K^{(i)}))_2 = \sum_{|\sigma|\leq 2} (D^\sigma v_K^{(i)},\ D^\sigma(q(v_K^{(1)})))_0$$

$$= \iiint_R v_K^{(i)} q(v_K^{(i)}) + q'(v_K^{(i)}) \sum_{1\leq|\sigma|\leq 2} |D^\sigma v_K^{(i)}|^2$$

since $q'' = 0$ a.e.

From our choice of m, $v_K^{(1)}q(v_K^{(1)}) + v_K^{(2)}q(v_K^{(2)}) > 0$ while the remaining terms are non-negative. Hence $q(v_K^{(1)}) - q(v_K^{(2)})$ is an admissible variation. Substituting in (7), we have:

$$\iiint_R f(x,y,t,v_K+w)[q(v_K^{(1)}) - q(v_K^{(2)})]\,dxdydt \leq 0 \quad .$$

By the mean value thoerem, we have:

$$\iiint_R [f(x,y,t,w) + f_u(x,y,t,\rho v_K+w)v_K][q(v_K^{(1)}) - q(v_K^{(2)})]\,dxdydt \leq 0$$

$$\text{for some} \quad 0 \leq \rho \leq 1 \quad .$$

Thus we obtain

$$\iiint_R f_u(\text{int. pt.})(v_K^{(1)} - v_K^{(2)})(q(v_K^{(1)}) - q(v_K^{(2)}))\,dxdydt$$

$$\leq - \iiint_R f(x,y,t,w)[q(v_K^{(1)}) - q(v_K^{(2)})]\,dxdydt$$

$$\leq \sup_R |f(x,y,t,w)| \iiint_R (|q(v_K^{(1)})| + |q(v_K^{(2)})|\,dxdydt \quad .$$

Using the monotonicity of q, $(v_K^{(1)} - v_K^{(2)})(q(v_K^{(1)}) - q(v_K^{(2)})) \geq 0$ so that the left-hand side is bounded above (using $f_u \geq \beta > 0$) by:

$$\beta \iiint_R (v_K^{(1)} - v_K^{(2)})(q(v_K^{(1)}) - q(v_K^{(2)}))\,dxdydt$$

$$= \beta \iiint_R [v_K^{(1)}q(v_K^{(1)}) + v_K^{(2)}q(v_K^{(2)})]\,dxdydt$$

$$\geq \beta m \iiint_R (|q(v_K^{(1)})| + |q(v_K^{(2)})|)\,dxdydt$$

$$\text{since} \quad \lambda q(\lambda) \geq m\,|q(\lambda)| \quad .$$

Thus $\beta m \leq \sup_R |f(x,y,t,w)|$.

But $\max_R |v_K| \leq \max_R |v_K^{(1)}| + \max_R |v_K^{(2)}| \leq 2M = 4m$.

Hence $\max_R |v_K| \leq 4m \leq 4/\beta \sup_R |f(x,y,t,w)|$.

We now consider a different variation. Denote by $\zeta^h(x,y,t)$, the time difference quotient

$$\zeta^h(x,y,t) \equiv \frac{1}{h}\,[\zeta(x,y,t+h) - \zeta(x,y,t)]$$

for any function ζ. We will show that the centered second difference $-(v_K^h)^{-h}$ is admissible.

$$(v_K, -(v_K^h)^{-h})_2 = (v_K(t), \frac{1}{h^2}\,(2v_K(t) - v_K(t-h) - v_K(t+h)))_2$$

$$= (\frac{v_K(t+h) - v_K(t)}{h}, \frac{v_K(t+h) - v_K(t)}{h})_2 \quad \text{by time periodicity}$$

$$= |v_K^h|_2^2 > 0 \quad \text{unless} \quad v_K^h \equiv 0, \quad \text{in which case the subsequent estimates are trivial.}$$

Thus by (7) again, we have:

$$\iiint_R f(x,y,t,v_K+w)[(-v_K^h)^{-h}]\,dxdydt \leq 0 \quad .$$

But the left-hand side is equal to ('integrating by parts')

$$\iiint_R (f(x,y,t,v_K+w))^h\, v_K^h\, dxdydt$$

and $(f(x,y,t,v_K+w))^h = f_t(\text{int. pt.}) + f_u(\text{int. pt.})(v_K+w)^h$. Thus

$$\iiint_R f_u(\text{int. pt.})(v_K^h)^2 dxdydt$$

$$\leq - \iiint (f_t(\text{int. pt.})v_K^h + f_u(\text{int. pt.})w^h v_K^h)\,dxdydt$$

which implies:

$$\beta|v_K^h|_0^2 \leq (|f_t(\text{int. pt.})|_0 + |f_u(\text{int. pt.})w^h|_0)\ |v_K^h|_0 \quad .$$

Letting $h \downarrow 0$, we have:

$$\beta|v_{K_t}|_0 \leq |f_t(\cdot,\cdot,\cdot,v_K+w)|_0 + \sup_R |f_u(\cdot,\cdot,\cdot,v_K+w)| \cdot |w_t|_0 \quad .$$

But $\phi \in N$ implies $|\phi_t|_0^2 = |\phi_x|_0^2 + |\phi_y|_0^2$. Thus

$$\beta\Big(\sum_{|\sigma|=1} |D^\sigma v_K|_0^2\Big)^{1/2} \leq 2(|f_t(\cdot,\cdot,\cdot,v_K+w)|_0 + \sup_R |f_u(\cdot,\cdot,\cdot,v_K+w)|\ |w_t|_0) \quad .$$

Now consider $\psi \equiv (((v_K^h)^{-h})^h)^{-h}$. By a calculation similar to the previous one,

$$(v_K, \psi) = |-(v_K^h)^{-h}|_2^2 > 0 \quad \text{(or else the estimates are trivial)} \quad .$$

Substituting ψ in (7), we obtain:

$$\iiint_R f(x,y,t,v_K+w)[((v_K^h)^{-h})^h]^{-h} dxdydt$$

$$= \iiint_R -\big([f(x,y,t,v_K+w)[^h\big)^{-h}(-(v_K^h)^{-h}) dxdydt \leq 0 \quad .$$

Now $(-f^h)^{-h}$ is a difference approximation to $(\frac{\partial}{\partial t})^2(-f)$ hence the mean value theorem implies

$$(-f^h)^{-h} = -f_{tt}(\cdot,\cdot,\cdot,\text{int. pt.}) - 2f_{tu}(\text{int. pt.})(v_K^h + w^h)$$

$$- f_{uu}(\text{int. pt.})(v_K^h + w^h)^2 + f_u(\text{int. pt.})(-(v_K^h)^{-h} - (w^h)^{-h}) \quad .$$

Thus

$$\beta|-(v_K^h)^{-h}|_0^2 \leq \iiint_R f_u(\text{int. pt.})(-(v_K^h)^{-h})^2 dxdydt$$

$$\leq \iiint_R \{[f_{tt}(\text{int. pt.}) + 2f_{tu}(\text{int. pt.})(v_K^h + w^h)$$

$$+ f_{uu}(\text{int. pt.})(v_K^h + w^h)^2 + f_u(\text{int. pt.})(w^h)^{-h})](-v_K^h)^{-h}\} dxdydt \, ,$$

whereupon we obtain

$$\beta|-(v_K^h)^{-h}|_0 \leq |f_{tt}(\text{int. pt.})|_0 + 2|f_{tu}(\text{int. pt.})(v_K^h + w^h)|_0$$

$$+ |f_{uu}(\text{int. pt.})(v_K^h + w^h)^2|_0 + |f_u(\text{int. pt.})(w^h)^{-h}|_0 \quad .$$

Letting $h \downarrow 0$, this becomes:

$$\beta|v_{Ktt}|_0 \leq |f_{tt}(\cdot,\cdot,\cdot,v_K+w)|_0 + 2|f_{tu}(\cdot,\cdot,\cdot,v_K+w)(v_{Kt} + w_t)|_0$$

$$+ |f_{uu}(\cdot,\cdot,\cdot,v_K+w)(v_{Kt} + w_t)^2|_0$$

$$+ |f_u(\cdot,\cdot,\cdot,v_K+w)w_{tt}|_0 \quad .$$

Using the wave equation $v_{Ktt} = \Delta v_K$, we obtain estimates on all the second order derivatives. This completes the proof of Theorem 3.

Theorem 2 now follows rather easily, since the inequalities of Theorem 3 imply that v_K is an interior minimum for K sufficiently large. In this case, relation (7) becomes an equality and all $\phi \in N \cap H^2$ are admissible. Since $N \cap H^2$ is dense in N, we have:

$$\iiint_R f(x,y,t,v_K+w)\phi \, dxdydt = 0 \tag{9}$$

for all $\phi \in N$. Moreover, v_K is unique, since if $\tilde{v}_K$ is another solution,

$$0 = \iiint_R [f(x,y,t,v_K+w) - f(x,y,t,\tilde{v}_K+w)](v_K - \tilde{v}_K)dxdydt$$

$$\geq \beta|v_K - \tilde{v}_K|_0^2 \quad ,$$

which is a contradiction. Using the estimates of Theorem 3 we obtain Theorem 2.

Greater regularity of v can be obtained by assuming more on f and w. In particular, suppose $f \in C^r$, $w \in H^r$ for some integer $r \geq 2$. Then we can obtain estimates for the H^r-norm of v and some pointwise bounds by the same methods as above. For instance, using the variation $q(v_t^{(1)}) - q(v_t^{(2)})$ with appropriate choice of cut-off point m, we obtain:

$$0 = \iiint_R [f_t(x,y,t,v+w) + f_u(x,y,t,v+w)(v_t+w_t)]$$

$$[q(v_t^{(1)}) - q(v_t^{(2)})]dxdydt \quad .$$

Thus

$$\iiint_R f_u(x,y,t,v+w)v_t(q(v_t^{(1)}) - q(v_t^{(2)}))dxdydt$$

$$\leq \iiint_R [-f_t(x,y,t,v+w) - f_u(x,y,t,v+w)w_t][q(v_t^{(1)}) - q(v_t^{(2)})]dxdydt$$

which, upon estimating as above, leads to:

$$\frac{\beta}{4} \sup_R |v_t| \leq \sup_R [|f_t(x,y,t,v+w)| + |f_u(x,y,t,v+w)w_t|] \quad .$$

Similarly, using higher order difference quotients in time, estimates of the form

$$|(\frac{\partial}{\partial t})^{\rho} v|_0 \le |(\frac{\partial}{\partial t})^{\rho} f|_0 + \rho |(\frac{\partial}{\partial t})^{\rho-1} \frac{\partial}{\partial u} f(v_t+w_t)|_0$$

$$+ \cdots + |f_u \cdot (\frac{\partial}{\partial t})^{\rho} w|_0 + |[(\frac{\partial}{\partial u})^{\rho} f](v_t+w_t)|_0$$

which, for $\rho > 3$, using the Sobolev inequality, and the wave equation readily imply that $v \in H^r$. Indeed, by this approach one proves:

<u>Theorem 4</u>. For $2 \le \rho \le r$, $|v|_{\rho} \le \hat{c}_{\rho}[|v|_{\rho-1} + |w|_{\rho} + 1]$ where $\hat{c}_{\rho}$ is a constant depending on f and its first ρ derivative and depending monotonically on $\|w\|_1$ and $\|v_t\|$.

For a full proof, see the appendix below.

3. Inverting the Linear Wave Operator

In solving the nonlinear wave equation (*), we will need to solve the linear wave equation:

$$w_{tt} - w_{xx} - w_{yy} = f(x,y,t) \tag{10}$$

for given $f \in H^r \cap N^{\perp}$, with w satisfying the boundary and periodicity conditions (1) above. The solution is easy using Fourier series.

Theorem 5. If $f \in H^r \cap N^{\perp}$, $\exists!\, w$

$$\Box w = f \quad \text{with} \quad w \in H^r \cap H^1 \cap N^{\perp} .$$

Proof. Construct a weak solution in $H^1 \cap N^{\perp}$ by Fourier series:

If
$$f = \sum_{\substack{k \\ 2k^2-j^2-\ell^2\neq 0}} \sum_{j>0} \sum_{\ell>0} f_{jk\ell}\, e^{\sqrt{2}ikt} \sin jx \sin \ell y$$

and
$$w = \sum_{\substack{k \\ 2k^2-j^2-\ell^2\neq 0}} \sum_{j>0} \sum_{\ell>0} w_{jk\ell}\, e^{\sqrt{2}ikt} \sin jx \sin \ell y$$

then clearly we must have

$$w_{jk\ell} = \frac{f_{jk\ell}}{-2k^2+j^2+\ell^2} .$$

The term $-2k^2 + j^2 + \ell^2$ is a non-zero integer hence $|(-2k^2 + j^2 + \ell^2)^{-1}| \leq 1$, which implies that

$$|w|_1 \leq |f|_1 \quad \text{and more generally}$$

$$|w|_\rho \leq |f|_\rho \quad \text{for} \quad 1 \leq \rho \leq r .$$

We note that unlike the one dimensional case, we gain no derivatives when solving (10). Indeed the sequence $j = k-1$, $\ell = k+1$ yields

$$-2k^2 + j^2 + \ell^2 = -2k^2 + k^2 - 2k+1 + k^2 + 2k+1 = 2$$

thus for large j, k, ℓ, $w_{jk\ell}$ decays no faster than $f_{jk\ell}$, at least for this sequence (and other obviously related ones).

Note that here our rationality conditions seem to be essential, since we have then good control on the norm of $\Box^{-1}$ on $N^{\perp}$.

For general domains or irrationally related side lengths, the spectrum of $-\Delta$ is less regular and a 'small divisor' problem arises.

4. Solution of the Non-linear Problem

We wish to solve (*): $\Box u + \varepsilon f(x,y,t,u) = 0$. A solution of the form:

$$u(x,y,t;\varepsilon) = v(x,y,t;\varepsilon) + \varepsilon w(x,y,t;\varepsilon)$$

will be constructed, where $v \in N$, $w \in N^{\perp}$. The method follows that of [6] and is by iteration.

Proof of Theorem 1: Let $u_0(x,y,t) = u(x,y,t;\varepsilon)$ where u_0 is determined by the bifurcation equation:

$$f(x,y,t,u_0) \in N^{\perp}.$$

Theorem 2 and 4 imply that for $f \in C^r$, $u_0 \in H^r$.

Let $w_1 \in N^{\perp}$ solve $\Box w_1 = -f(x,y,t,u_0)$. Theorem 5 implies $w_1 \in N^{\perp} \cap H^r$, and then, by Theorem 2 and 4 again, we can find $v_1 \in N \cap H^r$ s.t. $f(x,y,t,v_1+\varepsilon w_1) \in N^{\perp}$.

Continuing in this manner, suppose $u_{n-1} = v_{n+1} + \varepsilon w_{n-1}$ is known. Then solve $\Box w_n = -f(x,y,t,u_{n-1})$, obtaining $w_n \in N^{\perp} \cap H^r$, and then find $v_n \in N \cap H^r$ from the bifurcation equation $f(x,y,t,u_n) \in N^{\perp}$. If $\{u_n\}$ converges to u in a strong enough sense, we will have constructed a solution to (*).

To show the convergence of $\{u_n\}$, we follow [6] and first assume that $|u_n|_r$ is bounded. Define

$$\delta w_n = w_{n+1} - w_n;\ \delta v_n = v_{n+1} - v_n;\ \delta u_n = u_{n+1} - u_n = \delta v_n + \varepsilon \delta w_n \ .$$

Then $\Box \delta w_n = \Box w_{n+1} - \Box w_n = -[f(x,y,t,u_n) - f(x,y,t,u_{n-1})]$.

Hence
$$\begin{aligned}|\delta w_n|_0 &\le |f(x,y,t,u_n) - f(x,y,t,u_{n-1})|_0 \\ &= |f_u(x,y,t,\theta(x,y,t)u_n + (1-\theta)u_{n-1})\delta u_{n-1}|_0 \\ &\le \sup_R |f_u(x,y,t,\theta u_n + (1-\theta)u_{n-1})|\ |\delta u_{n-1}|_0 \ .\end{aligned}$$

Also, $f(x,y,t,u_n) - f(x,y,t,u_{n-1}) \in N^{\perp}$ so that

$$\iiint [f(x,y,t,u_n) - f(x,y,t,u_{n-1})]\,\delta v_{n-1}\,dxdydt = 0$$

$$= \iiint f_u(\text{int. pt.})[\delta v_{n-1} + \varepsilon\delta w_{n-1}]\,\delta v_{n-1}\,dxdydt \quad .$$

Thus

$$\beta|\delta v_{n-1}|_0 < \varepsilon \sup_R |f_u(\text{int. pt.})|\ |\delta w_{n-1}|_0$$

so that

$$|\delta u_n|_0^2 = |\delta v_n|_0^2 + \varepsilon^2|\delta w_n|_0^2$$

$$< \varepsilon^2\left(\frac{1}{\beta^2} \sup |f_u(\text{int. pt.})|^2 + 1\right) |\delta w_n|_0^2$$

$$< \varepsilon^2\left(\frac{1}{\beta^2} \sup |f_u(\text{int. pt.})|^2 + 1\right)(\sup |f_u(\text{int. pt.})|)^2|\delta u_{n-1}|_0^2$$

i.e. $|\delta u_n|_0 < \varepsilon\,(\frac{1}{\beta^2} B^2 + 1)^{1/2} B|\delta u_{n-1}|_0$ where $B \equiv \sup|fu(x,y,t,p)|$. If $|\varepsilon|(\frac{1}{\beta^2} B^2 + 1)^{1/2} B < 1$, $\delta u_n \to 0$ geometrically and thus $u_n \to u$ in H^0.

By interpolation, $\delta u_n \to 0$ in H^ρ for all $\rho < r$ since

$$|\delta u_n|_\rho < C_\rho\ |\delta u_n|_0^{1-\rho/r}|\delta u_n|_r^{\rho/r} \quad .$$

It thus only remains to show that $|u_n|_r$ is bounded.

For this, we require the following 'composition of functions inequality' [4]:

$$f \in C^k,\ u \in H^k \cap C^0 \quad ,$$

(11)

$$|f(x,y,t,u(x,y,t))|_k < \bar{C}_k(|u|_k + 1)$$

where $\bar{C}_k$ is a constant depending monotonically on $\|u\|_{C^0}$. Indeed, we prove inductively that there is an $\varepsilon_0 > 0$ such that, for some positive constants B, K, M,

$$\varepsilon_0|w_{n+1}|_r < M,\ \|v_n\|_r < K \quad .$$

If $\|v_{m-1}\| < B$, $|v_{m-1}|_r < K$, $\varepsilon_0|w_m|_r < M$ for all $m < n$ then by Theorem 2,

$$\|v_m\| < 4/\beta \sup_R |f(x,y,t,\varepsilon w_n(x,y,t))| \quad .$$

By Sobolev's inequality, $\|w_n\| < \|w_n\|_{C^1} < \alpha|w_n|_r < \alpha M$ for $|\varepsilon| < \varepsilon_0$. Thus,

defining $\chi(s) = 4/\beta \sup_{x,y,t;|p|<s} |f(x,y,t,p)|$ we have

$$\|v_n\| < \chi\left(\frac{\alpha M}{\varepsilon_0}\right) \quad .$$

From Theorem 4, $|v_n|_r < \hat{c}_r[\varepsilon_0\|w_n\|_1](|v_n|_{k-1} + \varepsilon_0 |w_n|_k + 1)$ for $|\varepsilon| < \varepsilon_0$.

Since $|v_n|_{r-1} < b\|v_n\|^{1/r}|v_n|_r^{1-1/r}$ by Sobolev (for r large enough) we have:

$$|v_n|_r < \hat{c}_r[\varepsilon_0\|w_n\|_1]\left(b\left(\frac{\|v_n\|}{|v_n|_r}\right)^{1/r} |v_n|_r + \varepsilon_0|w_n|_r + 1\right) \quad .$$

By considering the two cases where the factor on the right hand side multiplying $|v_n|_r$ is $<, > \frac{1}{2}$ respectively, we obtain

$$\begin{aligned} |v_n|_r &< 2\hat{c}_r[\varepsilon_0\|w_n\|_1](\varepsilon_0|w_n|_r + 1) \\ &< 2\hat{c}_r[\alpha M](M + 1) \end{aligned}$$

or

$$\begin{aligned} |v_n|_r &< (2\hat{c}_r[\varepsilon_0\|w_n\|_1]b)^r\|v_n\| \\ &< (2\hat{c}_r[\alpha M]b)^r\|v_n\| \quad . \end{aligned}$$

Adding, we have in all cases:

$$|v_n|_r < 2\hat{c}_r[\alpha M](M + 1) + (2\hat{c}_r[\alpha M]b)^r\|v_n\| \quad .$$

Finally, from Theorem 5 and the inequality (11)

$$\begin{aligned} |w_{n+1}|_r &< c_r|F(x,y,t,v_n + \varepsilon w_n)|_r < c_r\bar{c}_r[\|v_n+\varepsilon w_n\|](|v_n+\varepsilon w_n|_r + 1) \\ &< c_r\bar{c}_r[\|v_n\| + \alpha M](|v_n|_r + M + 1) \quad . \end{aligned}$$

Pick $\psi(s) > \max(\chi(s), 2\hat{c}_r[s], (2b\hat{c}_r[s])^r, c_r\bar{c}_r[s])$ monotone non-decreasing in s. We have derived the following inequalities:

$$\begin{cases} \|v_n\| < \psi(\alpha M) \\ |v_n|_r < \psi(\alpha M)(\|v_n + M + 1) \\ |w_{n+1}|_r < \psi(\|v_n\| + \alpha M)(|v_n|_r + M + 1) \end{cases} .$$

Pick $\alpha M = 1$ (determing M), choose $B = \psi(1)$ and $K > \xi(1)(B + M + 1) = B(B + M + 1)$.

Then the first two inductive steps are done and the final inequality becomes:

$$|w_{n+1}|_r < \psi(B+1)\cdot(K+M+1)$$

so choosing $\varepsilon_0 < M/\psi(B+1)\cdot(K+M+1)$ implies $\varepsilon_0|w_{n+1}|_r < M$.

To complete the proof, we note that these estimates hold to begin with: namely $w_0 = 0$, $v_0 = u_0$ is bounded in C^0 norm by $\chi(0)$ since $w_0 \equiv 0$ and $\chi(0) < \chi(1) < \psi(1) = B$. Similarly,

$$|v_0|_r < \psi(1)(B + M + 1) = K$$

and finally, $\varepsilon_0|w_1|_r < c_r\bar{c}_r[\|v_0\|](|v_0|_r + 1) < \psi(B)(M+1) < M$ by our choice of ε_0. This completes the proof of Theorem 1.

Note that in the proof, we utilized the Sobolev embedding theorem. Thus we must consider large enough values of r, namely those for which

$$\|w_n\|_{C^1} < \alpha|w_n|_r \quad \text{for some constant } \alpha \ .$$

In two space dimensions, this requires $r-1 > 3/2$ and for N-space variables, $r-1 > \frac{N+1}{2}$. If r is slightly larger $(r-2 > \frac{N+1}{2})$, then since our solution u belongs to H^ρ, $\rho < r$, we have a classical (C^2) solution.

5. Modifications for Other Domains, Periods, and Dimensions

The differences in considering

$$(*) \qquad u_{tt} - \Delta u + \epsilon f(x,t,u) = 0$$

with boundary and periodicity conditions on rectangular parallelopipeds with unequal, but rationally related sides and periods rationally related to the square roots of the sequence of eigenvalues of the Dirichlet Laplacian are rather minimal. The corresponding divisor when solving $\Box w = g$ for $w \in N^{\perp}$ is a rational number bounded away from zero, hence $\Box^{-1}$ is well-defined and bounded (perhaps by a constant larger than 1). Similarly, when using embedding theorems for several space variables, the assumed smoothness of f must be correspondingly higher. The estimates of Section 2 are identical, except that N_K must be a closed ball in the H^{ρ}-norm for $\rho > N+1/2$ and f must be C^{ρ}. The same trial variations of v_K will work, although more t-derivatives must be used if $N > 4$.

The result also extends to other domains and/or boundary conditions provided that the eigenvalues of $-\Delta$ satisfy the required rationality conditions. Thus, for instance, we could consider rectangular parallelopipeds with rational sides and impose periodicity in each x-coordinate separately or a combination of periodicity and Dirichlet boundary conditions. Of course, for several dimensions, the rationality restriction is more severe than in one space variable. What remains crucial is that $|\ell^2\omega^2 - \lambda_n| > \nu > 0$ for all $\ell > 0$ where λ_n are the eigenvalues of $-\Delta$, excluding those values ℓ, n for which this vanishes (the null space of).

We note also that the splitting of N into two orthogonal pieces - $\phi \in N \iff \phi = \phi^{(1)} - \phi^{(2)}$, which we used in making several key estimates, can be accomplished for any number of space variables (for an odd number of space dimensions, the expressions involve the imaginary parts of sums of exponentials instead of real parts, but are otherwise the same).

Appendix

Proof of Theorem 4: (See [4], [6].) Suppose $v \in H_r \cap N$ with $f(x,v+w) \in N^{\perp}$ for given $w \in H_r$ (here $x \in \Omega \subset \mathbb{R}^n$ and the time period is T). We wish to show that $|v|_r \le c_r[\|w\|_1, \|v_t\|](|v|_{r-1} + |w|_r + 1)$.

Since $v \in N$, it suffices to estimate

$$|\partial_t^r v| \le c_r[\|w\|_1, \|v_t\|](|v|_{r-1} + |w|_r + 1)$$

for then we have control over all r^{th} order derivatives, and lower order terms will be readily absorbed. Now $f \in N^{\perp}$, $u = v + w \in H_r \cap C$ implies $\partial_t^r f(x,y,t,v+w) \in N^{\perp}$. Let $f_t{}^\alpha$ denote α derivatives of f with respect to the third argument t, and similarly for f_u. Since $\partial_t^r v \in N$, we have:

$$0 = \iint_{\Omega\times[0,T]} \partial_t^r f(x,y,t,v+w) \cdot \partial_t^r v \, dxdydt$$

$$= \iint_{\Omega\times[0,T]} [(f_u(x,y,t,u)(\partial_t^r v + \partial_t^r w)\partial_t^r v] \, dxdydt$$

$$+ \iint_{\Omega\times[0,T]} [\partial_t^r f - f_u(\partial_t^r v + \partial_t^r w)](\partial_t^r v) \, dxdydt \quad .$$

Hence, by monotonicity and $f_u > \beta > 0$, we have:

$$\beta|\partial_t^r v|_0 \le |f_u(\partial_t^r v + \partial_t^r w) - \partial_t^r f|_0$$

$$+ \sup_{\substack{x,y,t\in\Omega,\, t\in[0,T] \\ |p|\le L}} |f_u(x,y,t,p)| \cdot |\partial_t w|_0$$

where L is the bound on $\sup u$ from the Sobolev inequality. Thus to complete the proof, we show, as in [6], that

$$|\partial_t^r f(x,t,u) - f_u \, \partial_t^r u|_0 \le c_r[\|w\|_1, \|v_t\|](|v|_{r-1} + |\phi|_{r-1} + 1)$$

which holds in general for C^r functions f. The difference consists of terms . the form $c_{\sigma\rho\alpha} f_{t^\sigma u^\rho}(\partial_t u)^{\alpha_1}(\partial_t^2 u)^{\alpha_2} \cdots (\partial_t^{r-1} u)^{\alpha_{r-1}}$ with

$\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1} = \rho$ and $r - \sigma = \alpha_1 + 2\alpha_2 + \cdots + (r-1)\alpha_{r-1}$ and $c_{\sigma\rho\alpha}$ some positive binomial type coefficients. Consider $f_{t^\sigma u^\rho} \prod_{j=1}^{r-1} (\partial_t^j u)^{\alpha_j}$ - let $u_0 = f_{t^\sigma u^\rho}$ and $u_j = \partial_t^j u$, $j = 1,\ldots,r-1$. For $j \geq 1$, we choose $p_j = \frac{(r-2)}{(j-1)\alpha_j}$.

Then $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{r-1}} = \frac{1}{r-2} \sum_{j=1}^{r-1} (j-1)\alpha_j = \frac{r-\sigma-\rho}{r-2}$. To use the Hölder inequality on the above product, we wish to choose $\frac{1}{p_0} = 1 - (\frac{r-\sigma-\rho}{r-2}) = \frac{\sigma+\rho-2}{r-2}$ which is fine for $\sigma + \rho \geq 2$. For $\rho = 0$, the given term is f_{t^r}, which we bound by a sup norm since u is bounded by assumption. $\begin{matrix}\rho = 1\\ \sigma = 0\end{matrix}$ leads to the only term we exclude in our sum, namely $f_u \partial_t^r u$. Thus all other terms satisfy $\sigma + \rho \geq 2$ and we choose p_0 as above. Then Hölder's inequality gives:

$$\iint_{\Omega\times[0,T]} \prod_{j=0}^{r-1} u_j^{2\alpha_j} \, dxdydt \leq \prod_{j=0}^{r-1} \left(\iint_{\Omega\times[0,T]} u_j^{2\alpha_j p_j} \, dxdydt\right)^{1/p_j} .$$

Applying the Gagliardo-Nirenberg inequality to a typical term in the product,

$$[\iint_{\Omega\times[0,T]} (\partial_t^j u)^{2(r-2)/(j-1)} \, dxdydt]^{(j-1)/2(r-2)} = \|\partial_t^j u\|_{L^{2(r-2)/j-1}}$$

$$\leq \text{const.} \ \|u_t\|_0^{1-(j-1/r-2)} |u|_{r-1}^{j-1/r-2} + |u_t|_0 \ .$$

But $|u_t|_0 \leq (\text{vol } R)^{1/2} \|u_t\|_0$ and $|u_t|_0 \leq |u|_{r-1}$, so

$$\left(\iint_{\Omega\times[0,T]} (\partial_t^j u)^{2(r-2)/(j-1)}\right)^{(j-1)/2(r-2)} \leq c\|u_t\|_0^{1-(j-1)/(r-2)} |u|_{r-1}^{(j-1)/(r-2)} .$$

Thus the product to be estimated

$$\prod_{j=0}^{r-1} \left(\iint_{\Omega\times[0,T]} u_j^{2\alpha_j p_j} dxdydt\right)^{1/p_j} \leq \prod_{j=1}^{r-1} \left(c\|u_t\|_0^{1-(j-1)/(r-2)} |u|_{r-1}^{(j-1)/(r-2)}\right)^{2\alpha_j}$$

$$\left(\iint_{\Omega\times[0,T]} u_0^{2\alpha_0 p_0}\, dxdydt\right)^{1/p_0}$$

$$= \left(\iint_{\Omega\times[0,T)} u_0^{2\alpha_0 p_0}\right)^{1/p_0} \cdot c^{2r} \|u_t\|_0^{2[\rho - \frac{(r-\sigma-\rho)}{r-2})} |u_{r-1}|^{\frac{r-\sigma-\rho}{r-2}}$$

since $\Sigma\alpha_j = r$, $\Sigma_j \alpha_j = r-\sigma$

$$< \left(\sup_{\Omega\times[0,T]} u_0\right) (T(\mathrm{vol}\ R))^{1/p_0} c^{2r}(\|u_t\|_0^{2r} + 1)(|u|_{r-1}^2 + 1)$$

since $\frac{r-(\sigma+\rho)}{r-2} < 1$.

Summing these terms with the proper coefficients leads to an estimate of the form:

$$|\partial_t^r f(x,t,u) - f_u(x,t,u)\partial_t^r u|_0 < c_r[\|u_t\|_0) \cdot (|u|_{r-1} + 1)$$

which proves Theorem 4.

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We consider the perturbed wave equation:

$$(*) \quad \begin{cases} u_{tt} - \Delta u + \varepsilon f(x,t,u) = 0 & \text{for } x \in \Omega \subset \mathbb{R}^n \\ & \quad t \in \mathbb{R} \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R} \end{cases}$$

where f depends monotonically on u and is periodic in t. Periodic solutions are constructed for ε sufficiently small under the following hypotheses:

(i) Ω is a domain such that the eigenvalues $\{\lambda_n\}$ of $-\Delta$ form a rational sequence - i.e.

(continued)

ABSTRACT (continued)

$$\lambda_k/\lambda_0 \in \mathbb{Q} .$$

(ii) The period of f is rationally related to the periods of the free vibrations for the wave equation, namely $\frac{2\pi}{\sqrt{\lambda_n}}$

(iii) f depends monotonically on u and is sufficiently smooth - $f \in C^r$ leads to a solution in H^ρ for $r > \frac{n+1}{2}$ and all $\rho < r$.

This generalizes a result of Rabinowitz [6] to more than one space variable.

END